# 1st Exercise Class For Mathematical Analysis

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# Personal Information

- My name's Yongli Peng, the TA for the class Mathematical Analysis (iii) to students in EECS, the instructor is Yuguang Shi (**ygshi@math.pku.edu.cn**).
- The classroom is at Building No.2, Room 317. The class time is on even Tuesday 18:40-20:30.
- I'm now a 1st-year graduate student in the school of Mathematical Science, PKU, majoring in the probability.
- My email is **yonglipeng@pku.edu.cn**. If you have any problem (problem about the homework, the exam and the class), you're welcome to send me a email (**I highly recommend you contact me in this way since I'll check my e-mail every day**).
- We will have a wechat group, where I'll send today's slides or notes, remember to download them before they become outdated.
- **You'd better not ask me questions in the wechat group because I may not check the notifications frequently.** But you can discuss problems with others in the group.

# Schedule

# Review: Basic concepts

# Review

- The concept of convergence, absolutely convergence, conditional convergence
- Whether we can commute the order of summation?
  the positive series can, the absolutely convergence can, but conditional convergence cannot (exa. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$).
- Cauchy's convergence test.
- Corollary: If $\sum_n a_n$ is convergent, then we have $\lim_{n\to\infty} a_n = 0$
- Some test methods (only for positive series): comparison test, D'Alembert's ratio test, Cauchy test, Raabe's test, the integral test for convergence
- The concept of the order:
  $1 < \log n < n^{\epsilon} < n < n\log n < n^{\frac{1}{\epsilon}} < e^n < n!(0 < \epsilon \ll 1)$

# Homework

# Homework

- HW for the last time: None.
- Today's HW (to be handed in): Exercise 9. 1. (1), (3), (5), (7); 2. (4) (6) (9) (17); 3, 6, 7. (1) (2); 8.

# Exercise

# 2. (6) Decide the convergence property

**Question1: 2. (6), Exer. 9**

Determine whether the positive series $\sum_{n=1}^{+\infty}\left[\frac{(1+1/n)^n}{\mathbf{e}}\right]^n$ converge?

**Proof.**

Calculate $\lim_{n\to\infty}\left[\frac{(1+1/n)^n}{\mathbf{e}}\right]^n$. Take logarithm and use Taylor's expansion: $\ln(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2}\frac{1}{n^2} + o(\frac{1}{n^2})$, we get

$$n^2\ln(1+\frac{1}{n}) - n = n - \frac{1}{2} + o(1) - n = -\frac{1}{2} + o(1) \to -\frac{1}{2},$$

$\lim_{n\to\infty}\left[\frac{(1+1/n)^n}{\mathbf{e}}\right]^n = -\frac{1}{2} \neq 0$. The series cannot converge. $\square$

# 2. (13) Decide the convergence property

**Question2: 2. (13), Exer. 9**

Determine whether the positive series $\sum_{n=2}^{+\infty} n^p \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right)$ converge?

**Proof.**

Just calculate the order: $\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} = \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n-1}\sqrt{n}} = \frac{1}{(\sqrt{n}+\sqrt{n-1})\sqrt{n-1}\sqrt{n}} \sim n^{-\frac{3}{2}}$.

So $n^p(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}) \sim n^{p-\frac{3}{2}}$, the series converge when $p < \frac{1}{2}$ and diverge otherwise. □

## 2. (16) Decide the convergence property

**Question3: 2. (16), Exer. 9**

Determine whether the positive series $\sum_{n=4}^{+\infty} \frac{1}{(\ln n)^{\ln(\ln n)}}$ converge?

**Proof.**

Take logarithm and we get: $\ln \frac{1}{(\ln n)^{\ln(\ln n)}} = -(\ln \ln n)^2$. Consider the limit
$\lim_{n\to\infty} \frac{(\ln \ln n)^2}{\ln n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x} = 0$ (L'Hospital's rule).
Then $\forall p > 0$, $\exists N > 0$, such that $(\ln \ln n)^2 < p \ln n$ is true for $\forall n > N$, which means
$\ln \frac{1}{(\ln n)^{\ln(\ln n)}} = -(\ln \ln n)^2 > -p \ln n = \ln \frac{1}{n^p}$ for $\forall n > N$.
So the series diverge.

□

# 2. (18) Decide the convergence property

**Question4: 2. (18), Exer. 9**

Determine whether the positive series $\sum_{n=1}^{+\infty} \frac{1-n\sin\frac{1}{n}}{n^{\alpha}} (\alpha > 0)$ converge?

**Proof.**

Use the Taylor's expansion for $\sin x$: $\sin x = x - \frac{1}{3!}x^3 + o(x^4)$, so
$\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{3!}\frac{1}{n^3} + o(\frac{1}{n^4})$ and $1 - n\sin\frac{1}{n} = \frac{1}{3!}\frac{1}{n^2} + o(\frac{1}{n^3})$.
$\frac{1-n\sin\frac{1}{n}}{n^{\alpha}} \sim \frac{1}{n^{\alpha+2}}$, so the series converge for $\alpha > 0$. Actually it converges for $\alpha > -1$ and diverges for $\alpha \leq -1$ $\square$

# 5. Proof

## Question5: 5, Exer. 9

If the function f(x) is defined on $[-1,1]$ and $f(0)=0$ with $f''(0)$ existing, then the series $\sum_{n=1}^{+\infty} f\left(\frac{1}{n}\right)$ converges if and only if $f'(0)=0$.

## Proof.

First we have $\lim_{n\to\infty} f(\frac{1}{n}) = f(0) = 0$ (f is continuous).
Then use the Taylor's expansion: $f(\frac{1}{n}) = f'(0)\frac{1}{n} + \frac{1}{2}f''(0)\frac{1}{n^2} + o(\frac{1}{n^2})$,
so $\forall\epsilon > 0$, $\exists N > 0$ such that $f(\frac{1}{n}) < f'(0)\frac{1}{n} + \frac{1}{2}f''(0)\frac{1}{n^2} + \epsilon\frac{1}{n^2}$ and
$f(\frac{1}{n}) > f'(0)\frac{1}{n} + \frac{1}{2}f''(0)\frac{1}{n^2} - \epsilon\frac{1}{n^2}$ remains true for $n > N$.
It follows that when $f'(0)=0$ the series must converge. And if the series converge and $f'(0)\neq 0$, then use the above inequality we can always derive that the series will diverge ($f'(0) < 0$ use the first inequality and $f'(0) > 0$ use the second one). □

# 9. Proof

**Question6: 9, Exer. 9**

If the positive series $\{a_n\}, \{b_n\}$ satisfy $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} (n \geq 1)$, then the convergence of $\sum_{n=1}^{+\infty} b_n$ implies the convergence of $\sum_{n=1}^{+\infty} a_n$

**Proof.**

Just use the comparison test. Since they are all positive, multiply them succesively we obtain $\frac{a_{n+1}}{a_1} \leq \frac{b_{n+1}}{b_1}$, which is just the condition for the comparison test. □

# Something I want to mention...

There's something I want to mention further, which are questions students asked after the class and I think it's important.

- One student asked the homework 2. (17), where I think it's natural to use the Stirling's Formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. This is an important formula in mathematical analysis (i), and I hope you can remember it. And it's okay to use it directly in the exam.
- For more details for this formula, check wiki at https://en.wikipedia.org/wiki/Stirling%27s_approximation.
- Another student asked me about the convergence of $\sum_n (-1)^n \frac{\sin n}{n}$. First you can use $|\sin n| > sin^2 n$ to check $\sum_n \frac{sinn}{n}$ is not absolutely convergent. And use the Dirichlet's test to check $\sum_n \frac{\sin n}{n}$ is convergent (Note $|\sum_{k=1}^n \sin k| = \frac{|\cos\frac{1}{2} - \cos\frac{n+1}{2}|}{|2\sin\frac{1}{2}|} \leq \frac{1}{|\sin\frac{1}{2}|}$ is bounded). So the series $\sum_n \frac{\sin n}{n}$ is conditional convergent.
- As for the series $\sum_n (-1)^n \frac{\sin n}{n}$, you can also use the Dirichlet's test. When n is even, for instance, let $n = 2m$, $|\sum_{k=1}^{2m} (-1)^k \sin k| \leq |\sum_{k=1}^m \sin 2k| + |\sum_{k=1}^m \sin(2k-1)|$, which is bounded similarly. So the series $\sum_n (-1)^n \frac{\sin n}{n}$ is also conditional convergent.

# Some jokes for fun...

塞德里克. 维拉尼 (Cedric Villani) 被昵称为 “数学界的摇滚巨星” —飘逸的长发、硕大的领花和蜘蛛胸针，常常引来旁人讶异的目光。他在 2010 年凭借对非线性朗道阻尼的证明以及 Boltzmand 方程的研究得到数学界的诺贝尔奖—菲尔兹奖。这是摘自他的自传《一个定理的诞生（我与菲尔兹奖的一千个日夜)》的一个小故事。

这让我回想起那场皮加勒乐队的演唱会，一曲疯狂的 Pogo 舞让一位可爱的朋克女孩突然撞进我怀里。我还记得她打满耳钉和唇环、活力四射的样子。

" 您的蜘蛛很漂亮。 "

“是啊，我总是佩戴一只蜘蛛，这是我的风格。这是我里昂请蜻蜓工艺坊的师傅为我专门设计的。”

“您是音乐家？”

“不是。”

“艺术家？”

“我是数学家。”

“什么？数学家？”

" 对... 就是数学家。 "

" 您研究什么？ "

" 嗯，您真的想知道？ "

“对啊，为什么不呢？”

“好吧，可别笑话我啊！”

我深吸了一口气。“我发展了一个关于完备局部紧度量空间上的里奇曲率的下界的综合性概念。”

# Some jokes for fun...

"什么?!"
"这是开玩笑吧?"
"这可不是玩笑。我这篇论文在圈内引起不小的反响呢!"
"您能重复一遍吗? 这太牛了!"
"好，我再说一遍：我发展了一套综合性理论，用于估计可分、完备并且局部紧的可测度量空间上的里奇曲率的下界."
"哇哦!"
"它是干什么用的?"
于是，话匣子就这样打开了。我开始慢慢讲解，耐心做了知识普及。爱因斯坦的相对论，是光线弯曲的曲率。曲率，非欧几里得几何的基石。当曲率为正时，光线相互靠近；当曲率为负时，光线发散。曲率这一光学概念，也与统计物理中的概念相结合，如密度、熵、动能、极小能量... 这是我和别人一起完成的发现。如何在一个像刺猬一样不光滑的空间上讨论曲率呢？最优运输，一个涉及工程学、气象学、计算机科学和几何学的概念。我那本上千页的书。我侃侃而谈，不知不觉走过来许多公里的路程。