2nd Exercise Class For Mathematical Analysis III

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Today's exercise and homework

For homework,

- Take your homework from the front desk.
- HW for the last time (to be returned): Exercise 9. 1.(1), (3), (5), (7); 2. (4) (5) (9) (17); 3; 7. (1), (2); 8
- HW this time (to be handed in): Exercise 9. 10. (2) (4) (6) (8), 11. (1) (3) (5), 13, 16, 17, 20, 21, 23, 24. (1) (4) (7), 26. Exercise 10. 2.(1) (3) (5) (7), 3. (2) (4) (6), 4, 5, 6, 7, 10, 11, 12, 13. (2), 14. (1)

For exercise,

- Exercise 9. 12, 14, 15;   Exercise 10. 8, 16, 17, 21
- (more tricky) Exercise 9. 19 (Hint: prove $x^t y^{1-t} \leq tx + (1-t)y$ for $x, y > 0$ and $0 \leq t \leq 1$), 18 (Hint: use 9. 19), 22. (Hint: the answer in the textbook) Exercise 10. 26 (Hint: cutoff (截断))
- (Ex. outside the book) If $\{a_n\}$ is decreasing, $\lim_n a_n = 0$ and the seq $\{\sigma_n = \sum_{k=1}^n (a_k - a_n)\}$ is bounded, then $\sum_n^\infty a_n$ converge.
  If $\{a_n\}$ is increasing and bounded, $a_n > 0$, then $\sum_n^\infty (1 - \frac{a_n}{a_{n+1}})$ converge.
  If $\{a_n\}$ is increasing and $\lim_n a_n = \infty$, then $\sum_n^\infty (1 - \frac{a_n}{a_{n+1}})$ diverges.

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  Since we only have the monotone property on $\{a_n\}$, you can expect nothing about $(2n+1)a_{2n+1}$ once you know the behavior on $\{2na_{2n}\}$. It's simple but you need to put it in your answer. Or you could just use the floor function, like $\lfloor \frac{n}{2} \rfloor a_n < \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n} a_i$

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- 7.(1) & 8 just need to use the Cauchy's convergence test, but you should write it carefully. It need $\frac{1}{S_n} \to 0$ or $r_n \to 0$.

Last time's HW: 2. (17)

2.(17)

Determine whether the series

$$\sum_{n=1}^{+\infty}\left[\frac{(2n)!!}{(2n+3)!!}\right]^{p}(p>0)$$

converge?

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String's formula: $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$ or
Wallis' formula: $\lim_{n\to+\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}=\frac{\pi}{2}$

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The order resulted is $O(n^{\frac{3}{2}})$.

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First use Raabe's test,

$$
\begin{aligned}
n(\frac{a_n}{a_{n+1}} - 1) &= n((\frac{2n+5}{2n+2})^p - 1) \\
&= n(1 + \frac{3p}{2n+2} + o(\frac{1}{n}) - 1) \\
&= \frac{3pn}{2n+2} + o(1) \to \frac{3p}{2}
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So we know it converges when $\frac{3}{2}p < 1$ and diverges when $\frac{3}{2}p > 1$ by Raabe's test.

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But when $p = \frac{2}{3}$ should be discussed alone.

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There can be two methods to discuss it, which is copied from your answers. The main argument is to prove

$$n(\frac{a_n}{a_{n+1}} - 1) < 1$$

is true when n is large.

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Method 1. Since $n(\frac{a_n}{a_{n+1}} - 1) = \frac{3pn}{2n} + \frac{9p(p-1)n}{2(2n+2)^2} + o(\frac{1}{n}) = \frac{2n}{2n+2} - \frac{n}{4(n+1)^2} + o(\frac{1}{n})$, so we can get $n(\frac{a_n}{a_{n+1}} - 1) < \frac{n-\frac{1}{8}}{n+1} + o(\frac{1}{n})$. For any $\epsilon > 0$, there exists $N > 0$ s.t. $\forall n > N$, $o(\frac{1}{n}) < \frac{\epsilon}{n}$ and $n(\frac{a_n}{a_{n+1}} - 1) < \frac{n-\frac{1}{8}+\epsilon}{n+1} < 1$.

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Method2. Use the Bernoulli's inequality: $(1+x)^\alpha < 1 + \alpha x$ when $0 < \alpha < 1$ and $(1+x)^\alpha < 1 + \alpha x$ when $\alpha > 1$.
$(1 + \frac{3}{2n+2})^p < 1 + \frac{3p}{2n+2}$ when $p < 1$.
So when $p = \frac{2}{3}$, $n(\frac{a_n}{a_{n+1}} - 1) < \frac{2n}{2n+2} < 1$.

Last time's HW: 7. (1)

7.(1)

If $\sum_n a_n$ diverge and $a_n > 0$, $S_n = a_1 + a_2 + \cdots + a_n$, then $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ diverge.

Proof.

Use the Cauchy's convergence test (you can also prove by contradiction), we just need to prove $\exists \epsilon > 0, \forall N > 0, \exists n > N, p > 0$, s.t.

$$\sum_{i=n+1}^{n+p} \frac{a_n}{S_n} > \epsilon$$

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Since $\sum_{i=n+1}^{n+p} \frac{a_i}{S_i} \geq \frac{\sum_{i=n+1}^{n+p} a_i}{S_{n+p}} = \frac{S_{n+p} - S_n}{S_{n+p}} = 1 - \frac{S_n}{S_{n+p}}$ and $S_n \to \infty$, so for fixed n, you can let p be large enough and obtain $\frac{S_n}{S_{n+p}} < \epsilon'$ $(\epsilon' < 1)$. Then

$$\sum_{i=n+1}^{n+p} \frac{a_n}{S_n} > 1 - \epsilon'$$

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$$\sum_{i=n+1}^{n+p} \frac{a_n}{S_n} > 1 - \epsilon'$$

Finally you just let $\epsilon = 1 - \epsilon'$ and $\forall N$ just let $n = N + 1$. □

Last time's HW: 8

8

If $\sum_n a_n$ converge, $a_n > 0$, $r_k = \sum_{n=k}^{\infty} a_n$, then $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverge.

Proof.

The proof is similar to the previous one.

More precisely, we have

$$\sum_{i=n}^{n+p} \frac{a_i}{r_i} \geq \frac{\sum_{i=n+1}^{n+p} a_i}{r_n} = \frac{r_n - r_{n+p+1}}{r_n} = 1 - \frac{r_{n+p+1}}{r_n}$$

When $n \to \infty$, we have $r_n \to 0$. So for any fixed n, we can let p be large enough such that $\frac{r_{n+p+1}}{r_n} < \epsilon'$.

This means $\forall n > 0, \exists p > 0$ such that $\sum_{i=n}^{n+p} \frac{a_i}{r_i} > \epsilon$ for a fixed constant $\epsilon = 1 - \epsilon'$.

By Cauchy's convergence test, we have $\sum_{i=1}^{\infty} \frac{a_i}{r_i}$ diverge. $\square$

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- $\prod a_n$ converge $\Longrightarrow \prod |a_n|$ converge. The other way is not right ($a_n = (-1)^n$).
- If $a_n > 0$, $\prod(1 + a_n)$ converge if and only if $\sum_n a_n$ converge.

Review: functional series

- The ordinary convergence in functional series is pointwise convergence: $\lim_n S_n(x) = S(x)$ for fixed x with $S_n(x)$ the partial sum.

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  `https://en.wikipedia.org/wiki/Arzela-Ascolitheorem`

  (The Lemma) Consider a sequence of real-valued continuous functions $\{f_n\}_{n\in\mathrm{N}}$ defined on a closed and bounded interval $[a, b]$ of the real line. If this sequence is *uniformly bounded* and *equicontinuous*, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathrm{N}}$ that converges uniformly. The converse is also true, in the sense that if every subsequence of $\{f_n\}$ itself has a uniformly convergent subsequence, then $\{f_n\}$ is uniformly bounded and equicontinuous.

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- Continuity preserving. Integrability preserving and integration calculation. Differetiablity preserving and calculation (for differentiation, we can only use the 1st derivative to control the original function, the converse is always impossible).

Exercise

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15

If $\sum_n a_n < \infty$ and $b_n = o(a_n)$, is it true that $\sum_n b_n$ must converge?

Proof

Proof.

All of the above deduction is true for positive series. So the counter-example should be given by general series, in which case the alternating series is one natural choice (this is the only one treated specially in the textbook, too general series have nothing different from the limit theory in Math.Anal I).
So the answer could be:

12. $a_n = (-1)^n \frac{1}{\sqrt{n}}$.

13. $b_n = a_n + o(a_n)$. $o(a_n)$ diverge when a_n converge, like $a_n = (-1)^n \frac{1}{\sqrt{n}}$ and $o(a_n) = \frac{1}{n}$.

14. Just use 13 and let $b_n = \frac{1}{n}$, $a_n = (-1)^n \frac{1}{\sqrt{n}}$ here. $\square$

Exercise

19. Holder's inequality

$p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\sum_n |a_n|^p$ and $\sum_n |b_n|^q$ converge. Prove $\sum_n a_n b_n$ absolutely converge and the Holder inequality:

$$\sum_n |a_n b_n| \leq (\sum_n |a_n|^p)^{\frac{1}{p}} (\textstyle\sum_n |b_n|^q)^{\frac{1}{q}}$$

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$$\sum_n |a_n b_n| \leq (\sum_n |a_n|^p)^{\frac{1}{p}(\sum_n |b_n|^q)^{\frac{1}{q}}}$$

18. Minkowski's inequality

$p > 0$, $\sum_n |a_n|^p$ and $\sum_n |b_n|^p$ converge. Prove the following inequality:

$$(\sum_n |a_n + b_n|^p)^{\frac{1}{p}} \leq (\sum_n |a_n|^p)^{\frac{1}{p}} + (\sum_n |b_n|^p)^{\frac{1}{p}}$$

Proof of the Holder

Proof.

For the Holder's inequality, we just need to take the limit of the finite Holder's inequality.

For the finite version, we need a auxiliary inequality:

$$x^t y^{1-t} \leq tx + (1-t)y, \quad \forall 0 \leq t \leq 1, x, y > 0 \tag{1}$$

Once this is proved, just let $x = \frac{|a_n|^p}{\sum |a_n|^p}$, $y = \frac{|b_n|}{\sum |b_n|^q}$ and $t = \frac{1}{p}$ (notice that $\frac{1}{p} + \frac{1}{q} = 1$). Then the inequality becomes:

$$\frac{|a_n|}{\left(\sum |a_n|^p\right)^{\frac{1}{p}}} \frac{|b_n|}{\left(\sum |b_n|^q\right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{|a_n|^p}{\sum |a_n|^p} + \frac{1}{q} \frac{|b_n|^q}{\sum |b_n|^q}$$

Finally we sum up the above inequality w.r.t. n and the right hand side becomes $\frac{1}{p} + \frac{1}{q} = 1$, which derives the Holder's inequality. □

Proof of the auxiliary inequality

Proof.

We can use the Jensen's inequality to prove the auxiliary inequality (1).

Jensen's inequality

For a real convex function φ , numbers $x_1, x_2, \dots, x_n$ in its domain, and positive weights a_i with $\sum_{i=1}^n a_i = 1$, Jensen's inequality can be stated as:

$$\varphi(\sum a_i x_i) \le \sum a_i \varphi(x_i),$$

where n can be finite of infinite (series). Moreover, this inequality can be generalized to integration (you can treat integration as uncountable summation where you summed up numbers indexed on the real line). The integral version is:

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \le \frac{1}{b-a}\int_a^b \varphi(f(x))dx$$

for any $f\colon [a,b] \to \mathrm{R}$ being a non-negative integrable function and φ is convex (note here the finite interval $[a,b]$ is necessary).

□

Proof of the auxiliary inequality

Proof.

Jensen's inequality (continuation)

More generally, rather than using uniform weights, we can use arbitrary weights in the integral version:

$$\varphi\left(\int_{\mathrm{R}} f(x)g(x)dx\right) \leq \int_{\mathrm{R}} \varphi(f(x))g(x)dx$$

where $f(x)$ is a non-negative integrable function, φ is convex and $g(x)$ is a non-negative function with $\int_{\mathrm{R}} g(x)dx = 1$.
In the previous case just let $g(x) = \frac{1}{b-a}1_{[a,b]}$ being constant on the interval $[a, b]$.
From https://en.wikipedia.org/wiki/Jensen%27s_inequality

Convex function

The original definition for a convex function is:
Let X be a convex set and a function $f\colon X \to \mathbf{R}$ is called *convex* if it satisfies:

$$\forall x_1, x_2 \in X, \forall t \in [0,1]: \quad f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

(from https://en.wikipedia.org/wiki/Convex_function)

Proof of the auxliary inequality

Convex function

There are some good properties for a convex function:

- A convex function f of one real variable is continuous (Lipschitz-continous) except on the endpoints,
  admits left/right derivatives and these derivatives are monotonically non-decreasing (this means at most countable points are indifferentiable).
- (star) A differentiable function of one is convex if and only if

  $$f(x) \geq f(y) + f'(y)(x-y),$$

  for multi-variable case $x, y \in \mathrm{R}^n$, this becomes

  $$f(x) \geq f(y) + \nabla f(y)^T (x-y),$$

- (star) A twice differentiable (C^2) function f of one variable is convex on an interval if and only if $f'' \geq 0$. For multi-variable case, the Hessian matrix ($\nabla^2 f$) shall be non-negative definite, that is, for any $u \in \mathrm{R}^n$, we have $u^T \nabla^2 f u \geq 0$.
- From `https://en.wikipedia.org/wiki/Convex_function`. The last two property is used more frequently.

  Remark: Convex optimization and convex analysis is particularly useful machine learning.

Proof of the auxliary inequality and Minkowski's inequality

Proof.

So first we can take the logarithm on both sides, which leads to

$$t \ln x + (1-t)\ln y \le \ln(tx + (1-t)y). \tag{2}$$

Taking the twice differentiation of $\ln x$, $(\ln x)'' = -\frac{1}{x^2} < 0$, so it's concave and $-\ln x$ is convex.

The Jensen's inequality tells us $-\ln(tx + (1-t)y) \le t[-\ln x] + (1-t)[-\ln y]$. So (2) is true and the auxiliary inequality follows from it.

Finally we attempt to prove the Minkowski's inequality using the Holder's inequality.

$$\begin{aligned}
\sum |a_n + b_n|^p &= \sum |a_n + b_n||a_n + b_n|^{p-1} \\
&\le \sum |a_n||a_n + b_n|^{p-1} + \sum |b_n||a_n + b_n|^{p-1} (\text{triangle's inequality}) \\
&\le \left(\sum |a_n|^p\right)^{\frac{1}{p}} \left(\sum |a_n + b_n|^{q(p-1)}\right)^{\frac{1}{q}} + \left(\sum |b_n|^p\right)^{\frac{1}{p}} \left(\sum |a_n + b_n|^{q(p-1)}\right)^{\frac{1}{q}} \\
&= \left(\sum |a_n|^p\right)^{\frac{1}{p}} \left(\sum |a_n + b_n|^{p}\right)^{1-\frac{1}{p}} + \left(\sum |b_n|^p\right)^{\frac{1}{p}} \left(\sum |a_n + b_n|^{p}\right)^{1-\frac{1}{p}} \\
&\quad \left(q(p-1) = p \text{ and } \frac{1}{q} = 1 - \frac{1}{p}\right)
\end{aligned}$$

Exercise

22

$a_n > 0$ and $\sum_n \frac{1}{a_n}$ converge. Prove the series $\sum_n \frac{n}{a_1+\cdots+a_n}$ converge.

Proof.

Use the answer in the textbook as a hint.
1. If $\{a_n\}_n$ is monotone, since $\sum_n \frac{1}{a_n}$ converge, we need $\frac{1}{a_n} \to 0$, which means $a_n \to \infty$. So we must have $\{a_n\}$ is increasing. In which case, we can have the following derivation:

$$\frac{n}{a_1 + a_2 + \cdots + a_n} \le \frac{n}{a_{\lfloor \frac{n}{2} \rfloor} + \cdots + a_n} \le \frac{n}{\frac{n}{2} a_{\lfloor \frac{n}{2} \rfloor}} = \frac{2}{a_{\lfloor \frac{n}{2} \rfloor}}$$

(if you use a_{2n} in this case, just like HW, you shall mention the odd case).
2. If $\{a_n\}$ is not monotone. Then let $b_1, b_2, \cdots, b_n$ be the rearrangement of the original $a_1, a_2, \cdots, a_n$ with $b_1 \le b_2 \le \cdots \le b_n$. For convenience we let $n = 2m$ here. Then we can derive:

$$\frac{n}{a_1 + a_2 + \cdots + a_n} = \frac{n}{b_1 + b_2 + \cdots + b_n} \le \frac{2}{b_m}, \tag{3}$$

where the last inequality follows from the case 1.

□

Proof of 22

Proof.

We sum the above inequality (3) up to $n = 2m$ and obtain:

$$\sum_{i=1}^{2m} \frac{i}{a_1 + a_2 + \cdots + a_i} \le \sum_{i=1}^{2m} \frac{2}{b_{\lfloor \frac{i}{2} \rfloor}} = \sum_{i=1}^{m} \frac{4}{b_i} \le \sum_{i=1}^{2m} \frac{4}{a_i},$$

where the intermediate equality is because almost every b_i is summed twice (you may need more subtle discussions of the beginning terms), and we need to relax the summation from m terms to $2m$ terms because we don't know where $b_1, b_2, \cdots, b_m$ lie in the original sequence $a_1, a_2, \cdots, a_n$. The safest way is to use all of them to control the summation of b_n.

Proof of 22

Proof.

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Remark: 1. Since $b_1, b_2, \cdots, b_n$ is just an rearrangement of the original $a_1, a_2, \cdots, a_n$, we cannot direct control the series $\sum_i^{\infty} \frac{i}{a_1+a_2+\cdots+a_i}$ but need to cope with the partial sum.
2. Since the above inequality is about the summation and we will take a limit afterwards, we don't need to discuss the odd case here.

Proof of 22

Proof.

We sum the above inequality (3) up to $n = 2m$ and obtain:

$$\sum_{i=1}^{2m} \frac{i}{a_1 + a_2 + \cdots + a_i} \le \sum_{i=1}^{2m} \frac{2}{b_{\lfloor \frac{i}{2} \rfloor}} = \sum_{i=1}^{m} \frac{4}{b_i} \le \sum_{i=1}^{2m} \frac{4}{a_i},$$

where the intermediate equality is because almost every b_i is summed twice (you may need more subtle discussions of the beginning terms), and we need to relax the summation from m terms to $2m$ terms because we don't know where $b_1, b_2, \cdots, b_m$ lie in the original sequence $a_1, a_2, \cdots, a_n$. The safest way is to use all of them to control the summation of b_n.

Remark: 1. Since $b_1, b_2, \cdots, b_n$ is just an rearrangement of the original $a_1, a_2, \cdots, a_n$, we cannot direct control the series $\sum_i^\infty \frac{i}{a_1+a_2+\cdots+a_i}$ but need to cope with the partial sum.
2. Since the above inequality is about the summation and we will take a limit afterwards, we don't need to discuss the odd case here.

Finally we let $m \to \infty$ and get $\sum_i^\infty \frac{i}{a_1+a_2+\cdots+a_i} \le 4 \sum_i^\infty \frac{1}{a_i} < \infty$. □

Exercise

10.8

If $f_n(x) \rightrightarrows f(x)$ and $g_n(x) \rightrightarrows g(x)$ on some interval I, $\{f_n\}$ and $\{g_n\}$ are uniformly bounded, then on I we have $f_n(x)g_n(x) \rightrightarrows f(x)g(x)$

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10.16

Prove the functional sequence $f_n(x) = (1 + \frac{x}{n})^n$ uniformly converge in $[0, 1]$ and calculate the limit:

$$\lim_{n\to\infty} \int_0^1 (1 + \frac{x}{n})^n dx$$

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17

$f_n \in R[a,b]$ and $f_n(x)$ inner closed uniformly converge to $f(x)$ on R, there is a function $g(x)$ satisfying $|f_n(x)| \le g(x)$ and $\int_R g(x) < \infty$. Prove $f(x) \in R(-\infty,\infty)$ and

$$\lim_n \int_R f_n(x)dx = \int_R \lim_n f_n(x)dx = \int_R f(x)dx$$

Proof of 10.8

Proof.

10.8. We just use the cutoff(截断) trick:

$$\begin{aligned}|f_n(x)g_n(x)-f(x)g(x)| &= |f_n(x)g_n(x)-f(x)g_n(x)+f(x)g_n(x)-f(x)g(x)| \\ &\le |f_n(x)-f(x)||g_n(x)|+|f(x)||g_n(x)-g(x)|\end{aligned}$$

Since f_n, g_n are uniformly bounded, we assume $\exists M$ s.t. $|f_n(x)|\le M$ and $|g_n(x)|\le M$. Let $n\to\infty$ and we can get $|f(x)|\le M$ and $|g(x)|\le M$ (this only needs pointwise convergence).
Then by the uniform convergence, $\forall \epsilon>0, \exists N>0$ s.t. $|f_n(x)-f(x)|<\frac{\epsilon}{2M}$ and $|g_n(x)-g(x)|<\frac{\epsilon}{2M}$ for $\forall n>N$.
It follows that

$$|f_n(x)g_n(x)-f(x)g(x)|<\frac{\epsilon}{2M}M+\frac{\epsilon}{2M}M=\epsilon.$$

for any ϵ and $x\in I$. Let $\epsilon\to 0$ and we get $f_n(x)g_n(x)\rightrightarrows f(x)g(x)$. □

Proof of 10.16

Proof.

First we just take the limit for a fixed x and obtain $\lim_n(1+\frac{x}{n})^n = e^x$. Then it's reasonable to expect the functional sequence will also uniformly converge to e^x.

Since $[0,1]$ is a closed interval and e^x is continuous on it, it's also uniformly continuous on it, i.e., $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x_1, x_2 \in [0,1]$ and $|x_1 - x_2| < \delta$, then $|e^{x_1} - e^{x_2}| < \epsilon$.

So we can first take a logarithm $\ln f_n(x) = n \ln(1+\frac{x}{n})$ and test its convergence. By Taylor's expansion, we have

$$\ln f_n(x) = n(\frac{x}{n} + o(\frac{x}{n})) = x + xo(1)$$

By $|x| \leq 1$ is finite, $|\ln f_n(x) - x| = |x|o(1) \leq o(1) \to 0$ as $n \to \infty$. More formally, $\forall \epsilon', \exists N$ s.t. $\forall n > N$, we have $|o(x)| \leq |o(1)| < \epsilon'$.

Finally using the uniform continuity, we just let $\epsilon' = \delta$ above and derive $|f_n(x) - e^x| < \epsilon$ for $n > N$. □

Proof of 10.17

Proof.

First we prove the integration $\int_{\mathrm{R}} f(x)dx$ by Cauchy's convergence test. Since $|f_n(x)| \leq g(x)$, we have $|f(x)| \leq g(x)$ by limitation. Then $\forall \epsilon, \exists M > 0$ s.t. $\forall n, m > M$ $\int_{-n}^{-m} g(x)dx + \int_m^n g(x)dx < \epsilon$, which leads to $|\int_{-n}^{-m} f(x)dx + \int_m^n f(x)dx| < \int_{-n}^{-m} |f(x)|dx + \int_m^n |f(x)|dx < \epsilon$. So by Cauchy's convergence test, we have $\int_{\mathrm{R}} f(x) < \infty$.

$\lim_n \int_{\mathrm{R}} f_n(x)dx = \int_{\mathrm{R}} f(x)dx$ follows similarly:

$$|\int_{\mathrm{R}} f_n(x)dx - \int_{\mathrm{R}} f_n(x)dx| \leq \int_{-\infty}^{-M} |f_n(x)|dx + \int_{-\infty}^{-M} |f(x)|dx + \int_M^{\infty} |f_n(x)|dx + \int_M^{\infty} |f(x)|dx$$
$$+ \int_{-M}^{M} |f_n(x) - f(x)|dx$$

By the same discussion, $\forall \epsilon > 0, \exists M$ s.t. the first 4 terms above are all less than ϵ. The last term is controlled by the uniform convergence: we can find n large enough s.t. $|f_n(x) - f(x)| < \frac{\epsilon}{2M}$. So we have $|\int_{\mathrm{R}} f_n(x)dx - \int_{\mathrm{R}} f_n(x)dx| < 4\epsilon + \frac{\epsilon}{2M}2M = 5\epsilon$. Finally let $\epsilon \to \infty$ and we get $\lim_n \int_{\mathrm{R}} f_n(x)dx = \int_{\mathrm{R}} f(x)dx$. □

Exercise

10.21

(1) f is differentiable on I, $f'(x)$ is uniformly continuous on I. Prove $F_n(x) = n[f(x + \frac{1}{n}) - f(x)]$ uniformly converges on I.

(2) Prove $f_n(x) = n(\sqrt{x + \frac{1}{n}} - \sqrt{x})$ inner closed uniformly converge on $(0, +\infty)$ but not uniformly converge on it.

Exercise

10.21

(1) f is differentiable on I, $f'(x)$ is uniformly continuous on I. Prove $F_n(x) = n[f(x + \frac{1}{n}) - f(x)]$ uniformly converges on I.

(2) Prove $f_n(x) = n(\sqrt{x + \frac{1}{n}} - \sqrt{x})$ inner closed uniformly converge on $(0, +\infty)$ but not uniformly converge on it.

10.26

$f_n(x)$ is continuous on $[0, 1]$, and pointwisely converge to $f(x)$. Then $f_n(x) \rightrightarrows f(x)$ if and only if $f_n(x)$ are equicontinuous on $[0, 1]$.

Proof of 10.21

Proof.

10.21.(1)

Notice here $F_n(x)$ looks like the definition $\frac{f(x+h)-f(x)}{h}$ of the differentiation when you replace the increment h by $\frac{1}{n}$, so it shall converge to $f'(x)$.

Since $f'(x)$ exists, we can apply the Lagrange Mean Value Theorem and get $F_n(x) = f'(x + \frac{\theta}{n})$ where $\theta \in [0, 1]$ and depends on x. But we have the uniform continuity of $f'(x)$ on I, so when n is large enough s.t. $|\frac{\theta}{n}| \leq \frac{1}{n} < \delta$ (δ is the one used in the definition of the uniform continuity), we have $|F_n(x) - f'(x)| < \epsilon$. This implies $F_n(x) \rightrightarrows f'(x)$.

10.21.(2)

By the above discussion, we know the singularity only occurs at the endpoint, that is, 0 and ∞. So the counter-example should converge to these two values. By the Cauchy's convergence test, we need the example satisfies: $\exists \epsilon, \forall N$, we can find $n, m > N$ and $x \in (0, \infty)$ s.t. $|f_n(x) - f_m(x)| > \epsilon$.

We just choose $n > N, m = 2n$ and $x = \frac{1}{n}$, then $|f_n(x) - f_m(x)| = |(\sqrt{2} - 1)\sqrt{n} - (\sqrt{6} - 2)\sqrt{n}| \to \infty$ as $n \to \infty$. □

Proof of 10.26

Proof.

First we prove the necessity.
Since $f \in C[0,1]$, it's also uniformly continuous on it. So $\forall \epsilon', \exists \delta_0, N$ s.t. whenever $|x_1 - x_2| < \delta_0$ and $n > N$, we have $|f(x_1) - f(x_2)| < \epsilon'$ and $|f_n(x) - f(x)| \leq \epsilon'$ for any $x \in [0,1]$.
Therefore for any $n > N$, we have

$$\begin{aligned} |f_n(x_1) - f_n(x_2)| &\leq |f_n(x_1) - f(x_1)| + |f(x_1) - f(x_2)| + |f_n(x_2) - f(x_2)| \\ &< \epsilon' + \epsilon' + \epsilon' = 3\epsilon' \qquad \forall |x_1 - x_2| < \delta_0, n > N \end{aligned}$$

Since f_n is also continuous on $[0,1]$, they are uniformly continuous. Hence we can find $\delta_1, \delta_2, \cdots, \delta_N$ s.t. $\forall |x_1 - x_2| < \delta_i$ we have $|f_i(x_1) - f_i(x_2)| < \epsilon$. Finally we let $\epsilon = 3\epsilon', \delta = \min\{\delta_0, \delta_1, \cdots, \delta_N\}$ and we derive the equicontinuity of $\{f_n\}$ on $[0,1]$: whenever $|x_1 - x_2| < \delta$ we have $|f_n(x_1) - f_n(x_2)| < \epsilon$ for any $n \in \mathrm{N}$.

Then we prove the sufficiency.
Since $\{f_n\}$ is equicontinuous, we can find δ, such that whenever $|x_1 - x_2| < \delta$ we have $|f_n(x_1) - f_n(x_2)| < \epsilon$ for any $n \in \mathrm{N}$. We can then give a finite partition of $[0,1]$: $0 = t_0 < t_1 < \cdots < t_k = 1$ with $|t_i - t_{i+1}| < \delta$. Since $[0,1]$ is finite, such a finite partition is possible. □

Proof of 10.26 (Continue)

Proof.

$\forall 0 \le i \le k$, since f_n pointwisely converge to $f(x)$, by Cauchy's convergence test, we can find N_i s.t. $\forall m, n > N_i$ we have $|f_n(t_i) - f_m(t_i)| < \epsilon$.

Since $k < \infty$, we can let $N = \max\{N_0, N_1, \cdots, N_k\}$ and obtain $N < \infty$. Then $\forall x$, it must lie in some interval, assume $t_i \le x < t_{i+1}$. Since $|x - t_{i+1}| \le |t_i - t_{i+1}| < \delta$. We have $|f_n(x) - f_n(t_{i+1})| < \epsilon$ for any $n \in \mathrm{N}$. So for any $n, m > N$, we have

$$\begin{aligned}|f_n(x) - f_m(x)| &\le |f_n(x) - f_n(t_{i+1})| + |f_n(t_{i+1}) - f_m(t_{i+1})| + |f_m(x) - f_m(t_{i+1})| \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon\end{aligned}$$

Finally by Cauchy's convergence test we obtain $f_n \rightrightarrows f$.

Remark: Actually you can also prove the uniform convergence directly without Cauchy, just combine the equicontinuous δ and f's uniformly continuous δ' togenther. $\square$

Exercise

exercise outside the book

If $\{a_n\}$ is decreasing, $\lim_n a_n = 0$ and the seq $\{\sigma_n = \sum_{k=1}^n (a_k - a_n)\}$ is bounded, then $\sum_n^\infty a_n$ converge.

Exercise

exercise outside the book

If $\{a_n\}$ is decreasing, $\lim_n a_n = 0$ and the seq $\{\sigma_n = \sum_{k=1}^{n}(a_k - a_n)\}$ is bounded, then $\sum_n^\infty a_n$ converge.

If $\{a_n\}$ is increasing and bounded, $a_n > 0$, then $\sum_n^\infty (1 - \frac{a_n}{a_{n+1}})$ converge.

Exercise

exercise outside the book

If $\{a_n\}$ is decreasing, $\lim_n a_n = 0$ and the seq $\{\sigma_n = \sum_{k=1}^{n}(a_k - a_n)\}$ is bounded, then $\sum_n^\infty a_n$ converge.

If $\{a_n\}$ is increasing and bounded, $a_n > 0$, then $\sum_n^\infty (1 - \frac{a_n}{a_{n+1}})$ converge.

If $\{a_n\}$ is increasing and $\lim_n a_n = \infty$, then $\sum_n^\infty (1 - \frac{a_n}{a_{n+1}})$ diverges.

proof

Proof.

1. Just notice somewhat tricky relation:

$$\sigma_n = \sum_{k=1}^{n}(a_k - a_n) = \sum_{k=1}^{n} a_k - na_n \geq \sum_{k=1}^{m} a_k - ma_n,$$

for $m < n$, where the last inequality follows from $\sum_{k=m+1}^{n} a_k \geq (n-m)a_n$. So $\sum_{k=1}^{m} a_k \leq ma_n + \sigma_n$. Since $\lim_n a_n = 0$ and σ_n bounded, when we let $n \to \infty$, we have $\sum_{k=1}^{m} a_k$ is bounded.
But from $\{a_n\}$ decreasing and $\lim_n a_n = 0$ we know this is a positive series and so we get its convergence.

2. Just notice

$$\sum_{n=m}^{M}(1 - \frac{a_n}{a_{n+1}}) \leq \frac{1}{a_m}(a_M - a_m) \leq \frac{\epsilon}{L-\epsilon}$$

where $\lim_n a_n = L$ (the existence follows from $\{a_n\}$ increasing and bounded), ϵ is chosen arbitrary. We know when m, M is large enough we can make $a_M - a_m < \epsilon$ and $a_m > L - \epsilon$ for any $\epsilon > 0$.
By Cahuchy's convergence test $\forall \epsilon'$ we just choose ϵ small enough such that $\frac{\epsilon}{L-\epsilon} < \epsilon'$ and then choose suitable threshold N. □

Proof

Proof.

Just notice:

$$\sum_{k=n}^{n+p}(1-\frac{a_n}{a_{n+1}}) \geq \frac{1}{a_{n+p+1}}(a_{n+p+1}-a_n) = 1-\frac{a_n}{a_{n+p+1}}$$

and when $p \to \infty$ with n fixed, we have $a_{n+p+p} \to \infty$ and the right hand side tends to 1.

So by Cauchy's convergence test, $\exists \epsilon$ fixed, $\forall N$ we just let $n = N+1$ and p large enough to make $1 - \frac{a_n}{a_{n+p+1}} > \epsilon$.

Thus the series diverge. □

Something I want to mention...

There's something I want to mention further, which are questions students asked after the class and I think it's important.

- Somebody asked me a question on the textbook: determine the convergence property of $\sum_{n=1}^{+\infty} \frac{(-1)^{[\sqrt{n}]}}{n^p} (p > 0)$.
  This can be done when you add up the terms with the same sign.

$$\sum_{n=4k^2}^{4k^2+8k+3} \frac{(-1)^{[\sqrt{n}]}}{n^p} = \sum_{n=4k^2}^{4k^2+4k} \frac{1}{n^p} - \sum_{k=4k^2+4k+1}^{4k^2+8k+3} \frac{1}{n^p} \leq \frac{4k+1}{4k^{2p}} - \frac{4k+3}{4(k+1)^{2p}}$$
$$= \frac{1}{k^{2p-1}} - \frac{1}{(k+1)^{2p-1}} + \frac{1}{k^{2p}} + \frac{1}{(k+1)^{2p}}$$

  So it's easy to prove the convergence of $\sum_{i=1}^{4k^2-1} \frac{(-1)^{[\sqrt{n}]}}{n^p}$ when $p > \frac{1}{2}$ if you add the above inequality up to k. This forms a convergent subsequence $\{S_{n_k}\}$ of the partial sum $\{S_n\}$. Since $|S_n - S_{n_k}| \leq \sum_{n=4k^2}^{4k^2+8k+3} \frac{1}{n^p} \leq \frac{8k+4}{4k^{2p}} \to 0$, this shows the convergence of the original sequence $\{S_n\}$.
  For $p \leq \frac{1}{2}$, by Cauchy's convergence test, $\sum_{n=4k^2}^{4k^2+4k} \frac{1}{n^p} \geq \frac{4k+1}{4(k+1)^{2p}}$ should approximate to 0 if it converge.
  But this is impossible, so it diverge when $p \leq \frac{1}{2}$.

Something I want to mention...

- For the absolute convergence, the critical value 1 is easily found and be verified.
- For 22, somebody said it can be done by a total rearrangement of $\{a_n\}$. I think it's okay to prove it in this way.