5th Exercise Class For Mathematical Analysis III

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Today's exercise and homework

For homework,

- Take your homework from the front desk.
- HW for the last time (to be returned): Exercise 11. 6. (3), (7), (11), 8, 9, 11, 14, 18
- HW this time (to be handed in): Exercise 12. 1. (1), (5), (7); 2. (3), (4); 3. (1); 6. (1), (2), (5); 8; 10; 12; 13
- And there's some HW hasn't be taken in the last two class. If you want it, remember to take it.

For exercise,

- Exercise 11. 12, 13, 15, 17
- Exercise 12. 4, 5, 9, 11, 14
- (more tricky) Exercise 9. 20.

Last time's HW

11.11

Assume the power series $f(x)=\sum_{n=1}^{+\infty} a_n(x-a)^n$ have convergence radius r, for $\forall b\in(a-r,a+r)$, we let $r'=\min\{b-a+r,a+r-b\}$. Prove

$$f(x)=\sum_{n=0}^{+\infty} b_n(x-b)^n,\quad x\in\left(b-r',b+r'\right)$$

where $b_n=\sum_{k=n}^{+\infty}\binom{k}{n}a_k(b-a)^{k-n}(n=0,1,2,\cdots)$.

Remark: It's easy to verify if $r'=\min\{b-a+r,a+r-b\}$, then $(b-r',b+r')\subset(a-r,a+r)$. But you cannot directly say it's (uniformly or absolutely) convergent and you can exchange the summation order, which is just the thing you need to prove! In order to exchange the order, you need something like absolutely convergence. Uniformly convergence is not enough, since you can treat a convergent number series as a functional series independent of x. Because it's convergent, it's also uniformly convergent as a functional series. But it's easy to find a counterexample to make the order not exchangeable. So you may need something like absoltely convergent.

Remark: One possible way is to use the theorem 11.1.1 in your textbook: if the power series $\sum a_nx^n$ converge at $x_0\neq 0$, it's inner closed absolutely uniformly convergent in the interval $(-|x_0|,|x_0|)$.

Proof

Proof.

More formally, for finite summation we have

$$\begin{aligned}\sum_{n=1}^{N} a_n(x-a)^n &= \sum_{n=1}^{N} a_n(x-b+b-a)^n \\ &= \sum_{n}^{N} a_n \sum_{k=0}^{n} (x-b)^k \binom{n}{k} (b-a)^{n-k} \quad (0 \le k \le n \le N, n \ge 1) \\ &= \sum_{k=0}^{N} \sum_{n=\max\{1,k\}}^{N} a_n (x-b)^k \binom{n}{k} (b-a)^{n-k}\end{aligned}$$

The terms are indexed in a 2 dimensional form, so you have different ways to add the terms up. With the above identity, the remaining part is just to prove the limitation can be taken: $N \to \infty$, which means the tail terms (from N to ∞) can tend to 0 as $N \to \infty$, no matter how you add the terms up. So a natural way to prove it is to ensure absolutely convergent. So you need to prove at any fixed $x \in (b-r', b+r')$, the series is absolutely convergent. □

Proof

Proof.

So the trick is, let $S(b.x) = \sum_{0\le k\le n}^{\infty} a_n(x-b)^k \binom{n}{k}(b-a)^{n-k}$ be a series, we want to prove its absolutely convergence, actually we can find x', b' such that $|x-b| = x'-b > 0$ and $b'-a = |b-a| > 0$. For instance, if $b < a, x < b$, let $b' = 2a-b$ and $x' = b' + (b-x) = 2a-x$. x' still lies in the convergent domain.

Use thm 11.1.1 we know $\sum_{n=1}^{\infty} a_n(x-a)^n$ is absolutely convergent at x' (choose x_0 a little large than x', $x'-a < x_0 - a < r$). Then its absolutely convergence implies the absolute convergence of $S(b', x')$ with 2 dimensional index. So we can exchange the order and prove the problem. □

Last time's HW

11.18

If $f(x)$ can be uniformly approximated by polynomials in (a, b), it's uniformly continuous in (a, b).

Proof.

- We just need to prove the limit $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist. Then make an extension $\tilde{f}$ of f from (a, b) to $[a, b]$: $\tilde{f} = f$ on (a, b) and $\tilde{f}(a) = \lim_{x\to a+} f(x), \tilde{f}(b) = \lim_{x\to b-} f(x)$.
- We have a uniform approximation $\{P_n\}$ of $f(x)$ in (a, b).
- First we need to prove the sequence $\{P_n(a)\}$ does have a limit, where we use the Cauchy criterion:
  $|P_n(a) - P_m(a)| \leq |P_n(a) - P_n(x)| + |P_m(a) - P_m(x)| + |P_n(x) - P_m(x)|$.
- Since $P_n(x)$ is continuous at a, $\forall \epsilon, \exists x > a, s.t. |P_n(a) - P_n(x)| < \epsilon$, so does $P_m(x)$. Then the first two terms is done.
- For the last term, use the uniformly convergence, we have $\{P_n\}$ is a uniform Cauchy's sequence in (a, b), which means $\forall x \in (a, b)$, we have $|P_n(x) - P_m(x)| < \epsilon$ for large enough n, m.
- Hence $\forall \epsilon, \exists N$, s.t. $\forall n, m > N$, we have $|P_n(a) - P_m(a)| < 3\epsilon$. This is just the Cauchy's criterion and so the sequence $\{P_n(a)\}$ converge.

Proof

Proof.

- $\{P_n(b)\}$ converge follows in the similar way as above. We assume $A = \lim_n P_n(a)$ and $B = \lim_n P_n(b)$.
- Then we want to prove $f(x) \to A$. In fact, we have: $|f(x) - A| \leq |f(x) - P_n(x)| + |P_n(x) - P_n(a)| + |P_n(a) - A|$.
- The middle term still follows the continuity of P_n at a. The first term is controlled by the uniform convergence of $P_n \to f$ in (a, b). The last term is what we have proved: $A = \lim_n P_n(a)$. So we have $f(x) \to A$ as $x \to a+$.
- Similarly we have $f(x) \to B$ as $x \to b-$. Finally make an extension $\tilde{f}$ of f as I mentioned. We have the uniform continuity of $\tilde{f}$. So $f(x)$ is uniformly continuous in (a, b).

□

Review. Power Series

- Thm 11.1.1: $\sum_n a_n x^n$ converges at x_0, then it's inner closed absolutely and uniformly convergent in $(-|x_0|, |x_0|)$.
- Thm 11.1.2: Let $\rho = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ and R is the convergent radius for power series $\sum_n a_n x^n$, then (1)If $\rho = \infty$, $R = 0$, (2)If $\rho = 0$, $R = \infty$, (3) If $0 < \rho < \infty$, $R = \frac{1}{\rho}$.
- Corollary: The above ρ can be replaced by $\rho = \lim_n \frac{|a_{n+1}|}{|a_n|}$ and we have similar results.
- Thm 11.2.1 (Abel): If the convergence radius for $\sum_n a_n x^n$ is R, then (1) it's inner closed uniformly convergent in $(-R, R)$.(2)If $\sum_n a_n R^n$ converges, the power series is uniformly convergent in any closed interval inside $(-R, R]$.(3)If $\sum_n a_n(-R)^n$ converges, the power series is uniformly convergent in any closed interval inside $[-R, R)$.
- $f(x) = \sum_{n=1}^{\infty} a_n(x-a)^n$ is continuous in convergence domain as a function, the infinite summation can exchange with the integral with respect to x. If the convergence radius is R, it can be differentiated arbitrary in $(x_0 - R, x_0 + R)$, i.e., it's smooth in that interval. So we have $a_n = \frac{f^{(n)}(x_0)}{n!}$.
- Weiestrass: for any function $f \in C[a, b]$, we can find a series of polynomials $\{P_n\}_{n\geq 1}$ s.t. $P_n \rightrightarrows f$ in $[a, b]$.

Review. Fourier Series

- The origin of Fourier series is
  $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0$ when $m \neq n$ and $= \pi$ when $m = n$.
  $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = 0$ when $m \neq n$ and $= \pi$ when $m = n$.
  $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0$ for any m, n.
  So if $f(x) = \frac{a_0}{2} + \sum_n a_n \cos(nx) + b_n \sin(nx)$, use the above identities we can get $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = a_n$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = b_n$. Actually, $\{sin(nx)\}$ and $\{cos(nx)\}$ form a orthogonal basis for function space, if you define the inner product to be $(f, g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$.
  For finite dimensional linear space, in linear algebra you have any vector can be represented as a linear combination of the basis. But in infinite dimensional space, you cannot always have this property. Even if you find a series of orthogonal functions, it's hard to justify that any function can be represented as a linear combination of those orthogonal ones.
- The actual meaning of above infinite linear combination is: you can find a series $\sum a_n e_n$ whose partial sum can approach any given function, where e_n are those orthogonal functions.
- A function space without any restriction is not possible to do the above things. One possible example is to consider functions with $\int f^2(x) < \infty$. Then $\{sin(nx)\}$ and $\{cos(nx)\}$ forms a possible series of orthogonal functions with the above property, if you consider the "approach" as convergence in the mean square: $g_n \to f$ means $\int |f - g_n|^2 dx \to 0$.

Review. Fourier Series

- The Fourier coefficients can be calculated in the previous slides. $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = a_n$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = b_n$ for any function. But the convergence needs to be proved, here you can only write $f \sim \frac{a_0}{2} + \sum_n a_n \cos(nx) + b_n \sin(nx)$.
- The Fourier coefficient and the Fourier series is not changed if you change finite values of the original function.
- (uniqueness) If f is continuous and 2π period, $a_n = b_n = 0$ implies $f = 0$.
- (Riemann-Lebesgue Lemma) If f is integrable on $[a, b]$ or absolutely integrable at its singular point in $[a, b]$, then $\int_a^b f(x)\sin(x)dx \to 0$ and $\int_a^b f(x)\sin(x)dx \to 0$ as $\lambda \to \infty$.
- (Riemann localizaion) The convergence of the Fourier series at x_0 only depends on the value of f in the neighborhood $(x_0 - \delta, x_0 + \delta)$ of x_0, $\forall \delta > 0$.

Review. The convergence of Fourier Series

- The main idea to prove the convergence is to use the Riemann-Lebesgue Lemma.
- Calculate the partial sum of the Fourier series when putting into the formula calculting the Fourier coeffients:

$$\begin{aligned}
S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\cos(nx)dt + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\sin(nx)dt \\
&= \cdots \\
&= \frac{1}{\pi}\int_{0}^{\pi} (f(x+t)+f(x-t))\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}dt
\end{aligned}$$

- So to prove $S_n \to S$, we just need to prove $\int_0^\pi \frac{f(x+t)+f(x-t)-2S_0}{2\pi\sin(\frac{t}{2})}\sin(n+\frac{1}{2})tdt \to 0$. Note $\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})t}{2\pi\sin(\frac{t}{2})}dt = 1$
- By Riem-Leb Lemma we have $\int_{-\pi}^{\pi}(\frac{1}{t} - \frac{1}{2\sin(\frac{t}{2})})\sin(n+\frac{1}{2})tdt \to 0$ as $n \to \infty$. So use the previous result and changing the variable we can obtain: $\int_R \frac{\sin(t)}{t}dt = \pi$.

Review. The convergence of Fourier Series

- So to use the Riemann-Lebesgue Lemma, we just need to justify when will the function $\frac{f(x+t)+f(x-t)-2S_0}{2\pi\sin(\frac{t}{2})}$ be well-behaved, that is, integrable or absolutely integrable at singular point.
- It's easy to see 0 is a possible singular point. So we need the function to have some great property to compensate this singularity at 0. Moreover, we need only to consider $\frac{f(x\pm t)-S_0}{t}$, since the singularity is cancelled for $\frac{1}{t}-\frac{1}{2\sin(\frac{t}{2})}$.
- Then various deduction can be referred to your textbook. Here just list the results ($\varphi(t)=f(x+t)+f(x-t)-2S_0$):
  - If f is piecewisely differentiable, we have $S_n(x)\to\frac{f(x+0)+f(x-0)}{2}$, where $f(x\pm 0)$ denotes the left (right) limit of f at x.
  - (Dini) f is integrable on $[-\pi,\pi]$ or absolutely integrable at singular point, $\forall x_0, \exists\delta>0, s.t. \int_0^\delta \frac{|\varphi(t)|}{t}dt<\infty$, then $S_n(x_0)\to S_0$.
  - (Lipschitz) f is integrable on $[-\pi,\pi]$ or absolutely integrable at singular point, $|f(x_0+t)-f(x_0)|\le L|t|^\alpha\ \forall t\in(x_0-\delta,x_0+\delta)$ ($\alpha-$Holder continuous), then $S_n(x_0)\to f(x_0)$. (Holder continuous is a generalization of Lipshitz continuous, and stronger than merely continuous, as we have mentioned, continuous is not enough to ensure the convergence of Fourier series, there's a counterexample).
  - (Dirichlet) f is integrable on $[-\pi,\pi]$ or absolutely integrable at singular point, if f is monotone on $(x_0-\delta,x_0)$ and $(x_0,x_0+\delta)$ where x_0 is not a singular point, then $S_n(x_0)\to\frac{f(x_0-0)+f(x_0+0)}{2}$.
  - More generally, piecewisely monotone is also sufficient (Jordan).

Review. The convergence of Fourier Series

- Sometimes we cannot prove the convergence, we will take a step back, consider some weaker form: does arithmatic average $\frac{S_0+S_1+\cdots+S_n}{n+1} \to f(x)$? Similarly replacing the coefficients and after tedious calculation, we get Fejer kernel $\Phi_n(t)$ and $\frac{S_0+S_1+\cdots+S_n}{n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\Phi_n(t)dt$.
- Typically you cannot use the previous knowledge to prove the convergence of Fourier series. This series is a special one. We not only want it converge, we also link the converging result with the original function. But 12.3.1 says the pointwise converging series with the original function f being continous and $2\pi-$periodic ensures the series converge to f.
- We can also consider the mean square convergence and the uniformly convergence of Fourier series, $\int |S_n - f|^2 dt$ and $S_n \rightrightarrows f$.
- The mean square convergence is the easiest to satisfy. If f is integrable on $[-\pi, \pi]$ or absolutely integrable at singular point, then $\int_{-\pi}^{\pi} |S_n - f|^2 dt \to 0$. As a corollary, we have the Parseval's identity: $\frac{a_0^2}{2} + \sum_n (a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$.
  And more genral Parseval's identity:
  $\frac{a_0\alpha_0}{2} + \sum_n (a_n\alpha_n + b_n\beta_n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$.
  Since it's derived from the mean square convergence, we don't need the series converge for above results.

- If f is $2\pi-$periodic and $f'(x)$ exists and integrable on $[-\pi,\pi]$, then $S_n(x) \rightrightarrows f(x)$.
- If f'' exists and integrable on $[-\pi,\pi]$, then we can differentiate termwise: $f'(x) = \sum_n (nb_n \cos(nx) - na \sin(nx))$. But the relation between the Fourier coefficients don't need $2-$order differentiation, just integrate by parts.
- If f is integrable on $[0,2\pi]$ and $2\pi-$periodic, then we can integrate termwise: $\int_0^x f(t)dt = \frac{a_0 x}{2} + \sum_n (\frac{a_n}{n}\sin(nx) + \frac{b_n(1-\cos(nx))}{n}0)$

Exercise

11.12

If the positive series $\sum_{n\geq 0} a_n$ diverge, the power series $\sum_{n\geq 0} a_n x^n$ has convergence radius 1, then $\lim_{x\to 1-0}\sum_{n=0} a_n x^n = \infty$.

Proof.

Let $f(x) = \sum_n a_n x^n$ be the power series, with domain $x \in (-1, 1)$. We want to prove $\lim_{x\to 1-0} f(x) = \infty$. So we need $\forall M > 0$ to find δ such that when $1 - x < \delta$ we have $f(x) > M$.

Since we have the inequality (when x is near 1, it must be positive and so each term in the series is positive):

$$
\begin{aligned}
f(x) = \sum_n a_n x^n &\geq \sum_{n=0}^{N} a_n x^n \\
&\geq \sum_{n=0}^{N} a_n (1-\delta)^n \quad \text{(each term is positive)} \\
&\geq \sum_{n=0}^{N} a_n (1 - n\delta) \quad \text{(Bernoulli's inequality)}
\end{aligned}
$$

$\square$

Proof

Proof.

Since $\sum_n a_n = \infty$ (a positive series can only be ∞ or converge), for the same M, we can find N_1, s.t. $\sum_{n=0}^{N_1} a_n > M + \epsilon$ for some fixed ϵ. Then for that N_1, $\sum_{n=0}^{N_1} na_n$ is finite, so we can find δ small enough so that $\delta < \frac{\epsilon}{\sum_{n=0}^{N_1} na_n}$.

Finally for the above inequality let $N = N_1$ and we obtain:

$f(x) \geq \sum_{n=0}^{N_1} a_n - (\sum_{n=0}^{N_1} na_n)\delta > M + \epsilon - \epsilon = M$. And this inequality is true for any $x > (1 - \delta)$, so this proves $\lim_{x \to 1-0} f(x) = \infty$. □

11.13

If $f(x)$ (not a constant) can be expanded as a power series at any point in $x_0 \in (a, b)$ with a positve radius $r > 0$. Prove the zero set of $f(x)$ in (a, b) does not have accumulation points.

Proof

Proof.

Since $f(x)$ can be expanded as a power series at any $x_0 \in (a, b)$ and f nonconstant, which implies the series cannot be $\equiv 0$. Otherwise we have $f \equiv 0$ on certain interval, then expand f as a power series at the endpoint of that interval we can extend the interval to get a larger interval on which $f \equiv 0$. Repeat this process and finally we can only have $f \equiv 0$ on (a, b), which contradicts our assumption.

Then if the zero set of f has a accumulation point x_0, we can derive the power series for f expanded at x_0 must be $f \equiv 0$, which contradicts our previous argument. To prove this, notice the power series expanded at x_0 is just the Taylor expansion of f at x_0, so we just need to prove the n-th differentiation $f^{(n)}(x_0) = 0$ at that point. Which can be done using mean value theorem.

First since x_0 is accumulation point, there is a seq of $\{x_n\}$ with $f(x_n) = 0$ and $x_n \to x_0$. Since f can be expanded as a power series at x_0, $f \in C^\infty$ near x_0. So it's continuous, and we have $f(x_0) = 0$.
Then for each $[x_{i-1}, x_i]$ (assume $x_{i-1} < x_i$ without loss of generality), since $f(x_{i-1}) = f(x_i) = 0$, we get a point $x_i^1 \in [x_{i-1}, x_i]$ with $f'(x_i^1) = 0$. This can be done for each i. For the inequality: $x_{i-1} \leq x_i^1 \leq x_i$ (or the converse direction) we have $|x_i^1 - x_0| \leq \max\{|x_i - x_0|, |x_{i-1} - x_0|\}$. Take a limit and we get $x_i^1 \to x_0$ with $f'(x_i^1) = 0$. Hence $f'(x_0) = 0$. □

Proof

Repeat the above process and we can get $\{x_i^2\}, \{x_i^2\}, \cdots$ with $f^{(2)}(x_0) = 0, f^{(3)}(x_0) = 0, \cdots$. So the expansion at x_0 must be identical to zero, which gives a contradiction.

11.15

If $f(x)$ is continuous on $[0,1]$ and satisfy:

$$\int_0^1 f(x)x^n dx = 0, \quad n = 0, 1, 2, \cdots$$

Prove: $f(x) \equiv 0$ on $[0,1]$.

Proof.

Since $\int_0^1 f(x)x^n dx = 0$ for any n, we get $\int_0^1 P_n(x)f(x)dx = 0$ for any polynomial P_n. But since f is continuous on $[0,1]$, use the Weiestrass approximation theorem and we get a seq of polynomials $\{P_n\}$ with $P_n \rightrightarrows f$. And so the limit can be taken into the integral and we get:

$$\lim \int f(x)P_n(x)dx = \int \lim f(x)P_n(x)dx = \int f(x)^2 dx$$

But the left hand side is always 0, so we get $\int f(x)^2 dx = 0$. Thus $f \equiv 0$ on $[0,1]$. □

Exercise

11.17

If f(x) can be approximated uniformly by polynomials on an infinite interval, prove $f(x)$ must be a polynomial too.

Proof.

Since $f(x)$ can be approximated uniformly by a series of polynomials. First we prove the degree of the polynomials must be equal as n is large enough. Then we prove each coefficient of order k will be unchanged as n is large enough. Combining these two results the limit $f(x)$ will only be a polynomial.

1. $degP_n$ must be unchanged. Suppose the infinite interval is just (b, ∞) without loss of generality. Then we have $|P_n - P_m| < \epsilon$ for $x > b$ and some $N > 0$ with $n, m > N$ by Cauchy's criterion. Then P_n and P_m must have the same order, otherwise when we let $|x| \to \infty$, $\frac{P_n}{P_m} \to \infty$ if $degP_n > degP_m$. But from $|P_n - P_m| < \epsilon$ we have $|\frac{P_n}{P_m} - 1| < \frac{\epsilon}{|P_m|} \to 0$ and so $|\frac{P_n}{P_m}|$ must be bounded, which is a contradiction.

2. In a similar way we can prove the coefficients of the same order must by equal for large n. From above discussion we have $\frac{P_n}{P_m} \to 1$ as $|x| \to \infty$. □

Exercise

Proof.

If we assume $degP_n = degP_m = d$, $P_n(x) = \sum_{i=0}^{d} a_i x^i$ and $P_m(x) = \sum_{i=0}^{d} b_i x^i$, then the above limit implies $\frac{P_n}{P_m} = \frac{a_d}{b_d} =$ as $|x| \to \infty$. So $a_d = b_d$ and this term is eliminated in $|P_n - P_m|$, which derives two $d-1$-th degree polynomial. Apply the above process repeatedly and in finite steps we can get $a_i = b_i$ for any $0 \leq i \leq d$.

Since the above steps only runs in finite steps, eventually we can get a finite threshold N s.t. $\forall n, m > N$, we have $P_n \equiv P_m$. Thus, f can only be a polynomial. $\square$

11.20

If f is continuous on $[0,1]$, $\forall n \in N$, define:

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

Prove: $B_n(f,x) \rightrightarrows f(x)$ on $[0,1]$ ($B_n(f,x)$ is called the Berstein polynomial of degree n, it gives a concrete polynomial approximation of any continuous function, which can be used in computation).

Exercise

Proof.

First notice $\sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} = 1$. In the probability, if you think x as the probability one trail succeeds. Then $\binom{n}{k} x^k(1-x)^{n-k}$ is just the probability that in n trials there is exactly k trials succeed. Then assume you are doing an experiment and you are observing the number of trials that succeed in these n trials. Once you get an observation k, you make a further calculation and obtain a quantity you are interested: $f(\frac{k}{n})$. So $\sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k}$ is just the averaged value you can get from the above experiment, observation and calculation. And $f(x)$ is actually the real value you desired, since $\frac{k}{n}$ you used in the calculation is just the frequency of the success trial, in probability we have $\frac{k}{n} \to x$ (frequency will approach probability) in some sense. Therefore $B_n(f,x) \rightrightarrows f(x)$ is natural.

To prove the argument formally, first we only need to prove $\sum_{k=0}^{n} \binom{n}{k} \left(f(\frac{k}{n}) - f(x)\right) x^k(1-x)^{n-k} \to 0$. But since f is continuous, we have a $\delta > 0$ s.t. if $|\frac{k}{n} - x| \le \delta$, we have $|f(\frac{k}{n}) - f(x)| < \epsilon$ and this part is done. The annoying part is the remaining one: $\sum_{|k-nx|>n\delta} \binom{n}{k} \left(f(\frac{k}{n}) - f(x)\right) x^k(1-x)^{n-k}$ □

Exercise

Proof.

A few calculation concerning combinatory gives: $\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k}=nx$ and $\sum_{k=0}^{n}\binom{n}{k}k(k-1)x^k(1-x)^{n-k}=x^2n(n-1)$. So we have $\sum_{k=0}^{n}\binom{n}{k}(\frac{k}{n}-x)^2x^k(1-x)^{n-k}=\frac{x(1-x)}{n}$, which is just the variance you learned in the high school. But this can help us to control the remaining terms:

$$\frac{x(1-x)}{n}=\sum_{k=0}^{n}\binom{n}{k}(\frac{k}{n}-x)^2x^k(1-x)^{n-k}\geq\sum_{|k-nx|>n\delta}\binom{n}{k}(\frac{k}{n}-x)^2x^k(1-x)^{n-k}$$
$$\geq[\sum_{|k-nx|>n\delta}\binom{n}{k}x^k(1-x)^{n-k}]\epsilon^2$$

So $\sum_{|k-nx|>n\delta}\binom{n}{k}x^k(1-x)^{n-k}\leq\frac{x(1-x)}{n\epsilon^2}\to 0$, which means for $\epsilon>0$ we can have n large enough so that $\sum_{|k-nx|>n\delta}\binom{n}{k}\left(f(\frac{k}{n})-f(x)\right)x^k(1-x)^{n-k}\leq 2M\frac{x(1-x)}{n\epsilon^2}\leq\epsilon$ where $M=\max_{x\in[0,1]}|f(x)|$. This completes the proof. □

Exercise

12.4

If $f(x)$ is $2\pi-$periodic and $f \in C^2(-\infty, \infty)$, assume the Fourier series of f is:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx).$$

Calculate the Fourier series for $f'(x)$ and $f''(x)$ and prove $\exists C > 0$ s.t.

$$|a_n| \leqslant \frac{C}{n^2}, \quad |b_n| \leqslant \frac{C}{n^2}, \quad n = 1, 2, \cdots$$

Proof.

The Fourier series for f' and f'' can be derived just from integration by parts: $f' \sim \sum(nb_n \cos(nx) - na_n \sin(nx))$ and $f'' \sim \sum(-n^2 a_n \cos(nx) - n^2 b_n \sin(nx))$. Since f'' is continuous and $2\pi-$periodic implies f'' is bounded, we have $|a_n''| = \frac{1}{\pi} | \int_{-\pi}^{\pi} f''(x) \cos(nx) dx| \leq \frac{1}{\pi} \int |f''(x)| dx \leq 2M$, where $M \geq |f''(x)|$. So $\exists C > 0$, s,t, $|a_n| \leqslant \frac{C}{n^2}, \quad |b_n| \leqslant \frac{C}{n^2}$. □

Exercise

12.5

If f can be expressed as the difference of two monotone functions on $[0, 2\pi]$, its Fourier series is denoted as:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) .$$

Prove: $a_n = O\left(\frac{1}{n}\right), b_n = O\left(\frac{1}{n}\right)$.

Proof.

We just need to prove the case when f is monotone. And we just assume f is decreasing. Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx = \frac{1}{n\pi}\int_0^{2n\pi} f(\frac{u}{n})\cos u du \quad (u = nx) \\ &= \frac{1}{n\pi}[\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} + \cdots] \\ &\le \frac{1}{n\pi}[f(\frac{\pi}{2n}) - f(\frac{\pi}{2n}) + f(\frac{\pi}{n}) - f(\frac{2\pi}{n}) + \cdots] \end{aligned}$$

□

Exercise

Proof.

Since $\int_{\frac{k\pi}{2}}^{\frac{(k+1)\pi}{2}} \cos u du = 1$ when k is even and $= -1$ when k is odd, use the monotone property we can derive the above alternating summation. To be precise, we get: $a_n \leq \frac{1}{n\pi}[f(\frac{\pi}{n}) - f(\frac{2\pi}{n}) + f(\frac{3\pi}{n}) - \cdots]$. On the one hand, since f is decreasing, we have $f(\frac{(2k-1)\pi}{n}) \geq f(\frac{2k\pi}{n})$, we have the right hand side $RHS \geq 0$. On the other hand, we have $f(\frac{2k\pi}{n}) \geq f(\frac{(2k+1)\pi}{n})$ and we get $RHS \leq \frac{1}{n\pi}(f(\frac{\pi}{n}) + f(2\pi))$. Hence we get $a_n = O(\frac{1}{n})$. The case for b_n follows in the similar way. □

12.9

If $f \in C^1[0, 2\pi]$ and $f(0) = f(2\pi)$, $\int_0^{2\pi} f(x)dx = 0$. Prove $\int_0^{2\pi} [f'(x)]^2 \mathrm{d}x \geqslant \int_0^{2\pi} [f(x)]^2 \mathrm{d}x$.

Proof.

Just use the Parseval's identity for $f(x)$ and $f'(x)$. Notice the relationship between the Fourier coefficients of f and f'. □

Exercise

12.11

Prove there exists infinitely many Fourier series $\sum_{n=0}^{+\infty}(a_n \cos nx + b_n \sin nx)$ uniformly converge to 0 on $[-1, 1]$.

Proof.

Just find functions equal 0 on $[-1, 1]$ but ensure their Fourier series uniformly converge. By theorems in the textbook we just need $f'(x) \in R[-\pi, \pi]$. A good choice is to make $f'(x)$ continuous. So we just need to extend 0 on $[-1, 1]$ to $[-\pi, \pi]$ so that their first order differentiation will be consistent at the endpoint 1 and -1. Various quadratic funtions may be good choice since their differentiation at the their extreme point is zero.

A special case is $e^{-\frac{1}{x^2}}$. This function is even and its arbitrary differentiaion at $x = 0$ is 0, but itself is not 0, which means we can not only require the first order differentiation to be consistent at the endpoint, but we actually can have any order differentiation to be consistent. $\square$

Exercise

12.14

Assume f, g are $2\pi-$periodic and integrable on $[-\pi, \pi]$. Prove their Fourier series are identical if and only if $\int_{-\pi}^{\pi} |f(x) - g(x)| \mathrm{d}x = 0$.

Proof.

Just use Parseval's identity. f and g have identical Fourier series is equivalent to $f - g$ has all Fourier coefficients to be zero, which indicates $\int |f-g|^2 dx = 0$. But first by Cauchy-Schwartz inequality, $\int_{-\pi}^{\pi} |f-g| dx \leq (\int_{-\pi}^{\pi} |f-g|^2 dx)^{0.5} (\int_{-\pi}^{\pi} dx)^{0.5}$. On the other hand, we have $\int_{-\pi}^{\pi} |f-g|^2 dx \leq 2M \int_{-\pi}^{\pi} |f-g| dx$ where $M > |f(x)|$ and $M > |g(x)|$ on $[-\pi, \pi]$ (Riemann integrable indicates the function is bounded).

Therefore $\int_{-\pi}^{\pi} |f-g|^2 dx = 0$ iff $\int_{-\pi}^{\pi} |f-g| dx = 0$ and this completes the proof. □